Teaching mathematical reasoning

The challenge of the new syllabus

Peter Gould, 2013

Education & Communities

Why focus on reasoning?

The widespread utility and effectiveness of mathematics come not just from mastering specific skills, topics, and techniques,

but more importantly, from developing the ***ways of thinking****—the habits of mind—used to create the results.*

Al Cuoco, Paul Goldenberg, & June Mark
"Habits of Mind: An Organizing Principle for High School Curricula."
The Journal of Mathematical Behavior, 1996.

What do we mean by reasoning?

…logical thought and actions, such as analysing, proving, evaluating, explaining, inferring, justifying, and generalising.

Mathematics K-10 Syllabus, p. 36

Reasoning

Students are reasoning mathematically when they explain their thinking, deduce and justify strategies used and conclusions reached, ... and explain their choices.

Mathematics K-10 Syllabus p. 36

Communicating reasoning

Students *explaining their thinking* is not necessarily (nor uniquely) reasoning mathematically.
Communicating mathematical reasoning may need its own teaching focus.

Writing up your reasoning is much easier if you can actually carry out the necessary reasoning.

Two difficult questions

1. *How do you teach students to reason mathematically?*
2. *How do you teach students to communicate their mathematical reasoning?*

Reasoning in the new syllabus

S2	S3	S4	5.1.3	5.2.3	5.3.3
check the accuracy of a statement and explains the reasoning used MA2-3WM					

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Communicating reasoning

Three steps in teaching students to communicate mathematical reasoning:

1. *Convince yourself*
2. *Convince a friend*
3. *Convince an enemy (or sceptic)*

A staff room debate: Exploring the syllabus

Choose one of the following, tell me what the answer is and how you worked it out:

$$\sqrt{0.4}$$

$$\sqrt{0.1}$$

$$\sqrt{0.09}$$

Is this a reasonable question for your students?

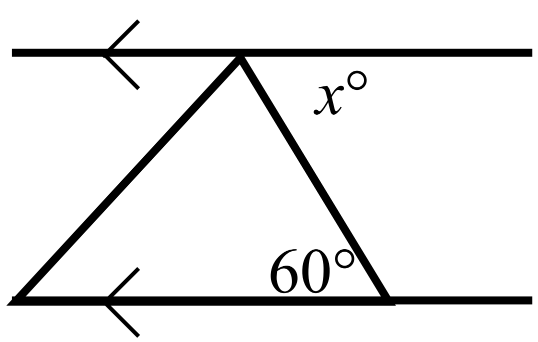
Is this a Stage 4 question?

- recognise the link between squares and square roots p. 278
- determine the two integers between which the square root of a non-square *whole number* lies (Reasoning) p. 279

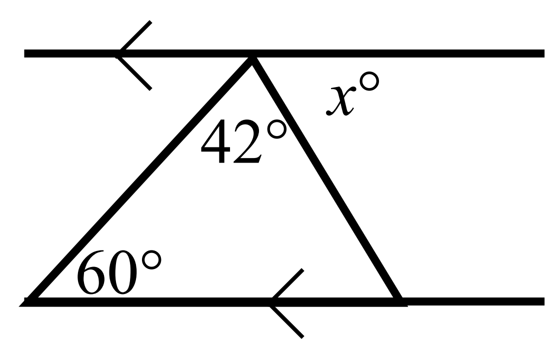
Teaching reasoning

Typically in Stage 4, we start with questions that require one-step of reasoning and progress through to two-steps of reasoning.
For example, students are expected to use their knowledge of parallel lines and congruent figures to solve numerical exercises on finding unknown lengths and angles in figures.

One-step of reasoning?



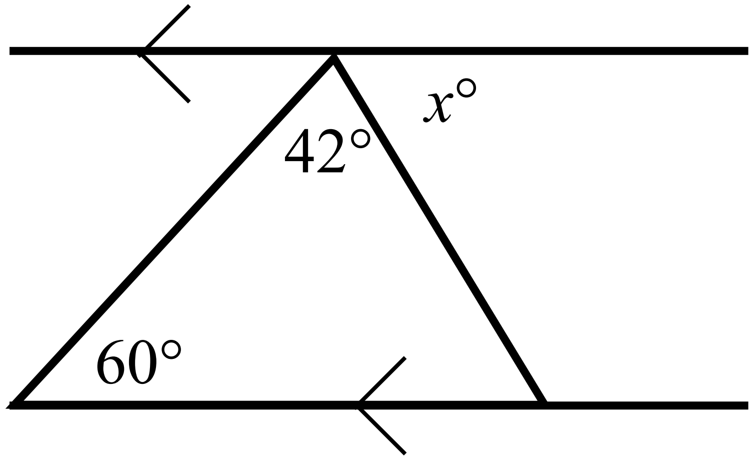
Two-steps of reasoning?



Teaching reasoning

As well as increasing the complexity of chains of reasoning we need to assist students to understand that multiple pathways of reasoning are possible in mathematics.

How many different ways are there to answer the following question?



Is there a most efficient way?

- compare different solutions for the same problem to determine the most efficient method (5.2)

Teaching reasoning

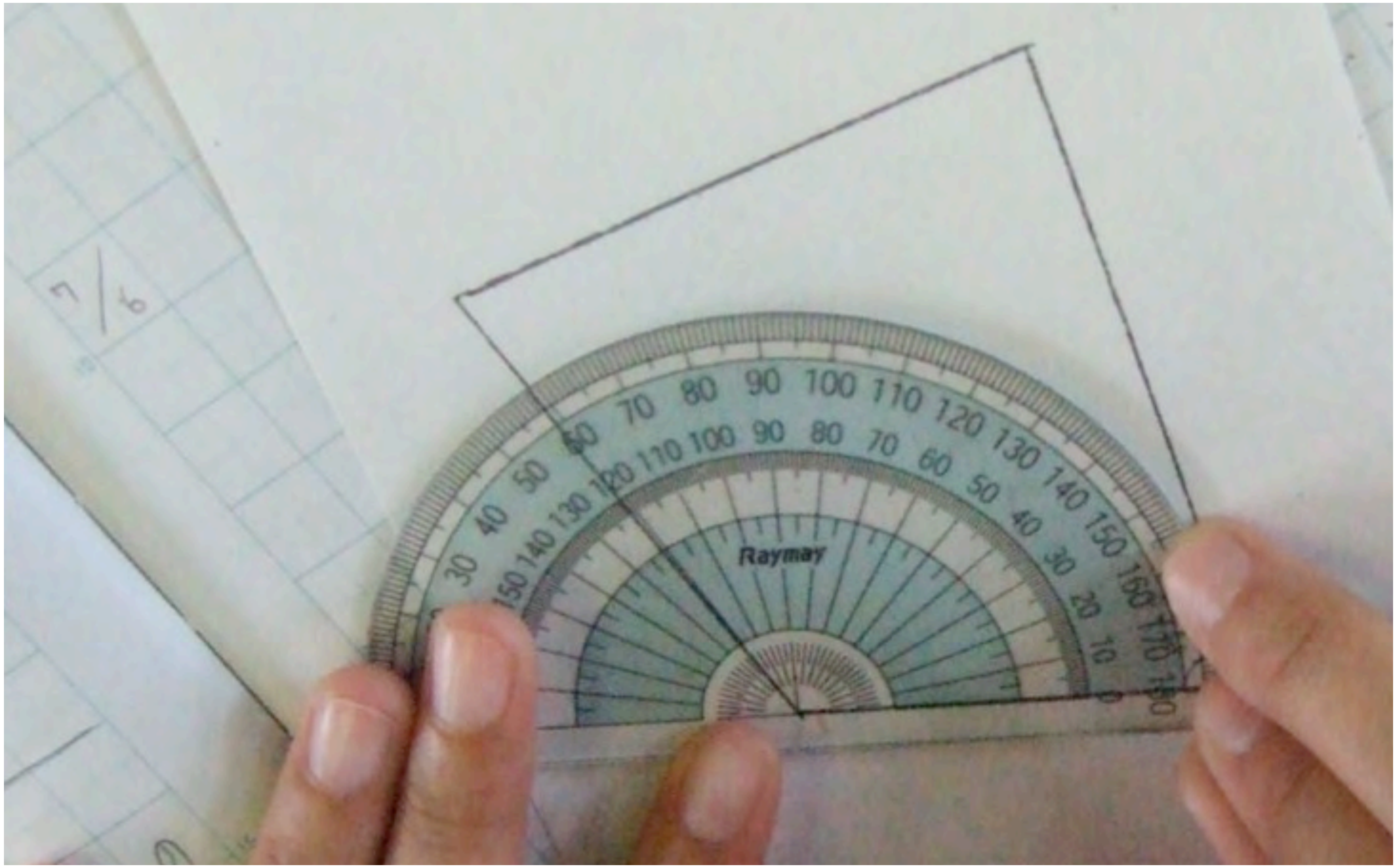
Students need to learn that ***multiple pathways of reasoning*** are possible.

- apply geometrical facts, properties and relationships to find unknown sides and angles in diagrams, providing appropriate reasons
  - ***recognise that more than one method of solution is possible*** (Reasoning)
  - compare different solutions for the same problem to determine the most efficient method (Communicating, Reasoning)

Mathematics Syllabus K-10, 5.2 p. 356

Explore: Using a protractor

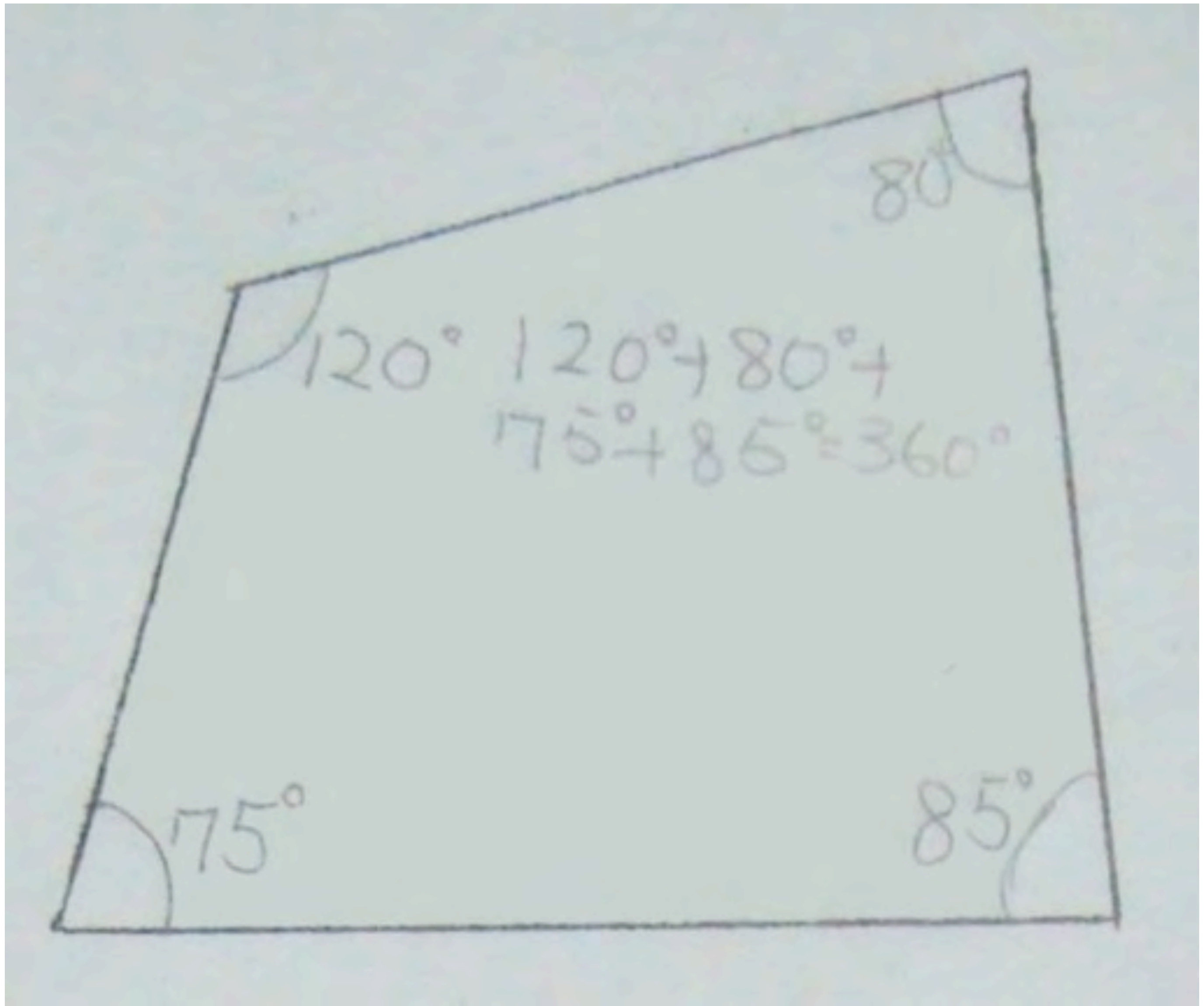
Demonstrate that the angle sum of a triangle is 180˚ and use this to find the angle sum of a quadrilateral. (Stage 4)



Convince yourself.

Exploring

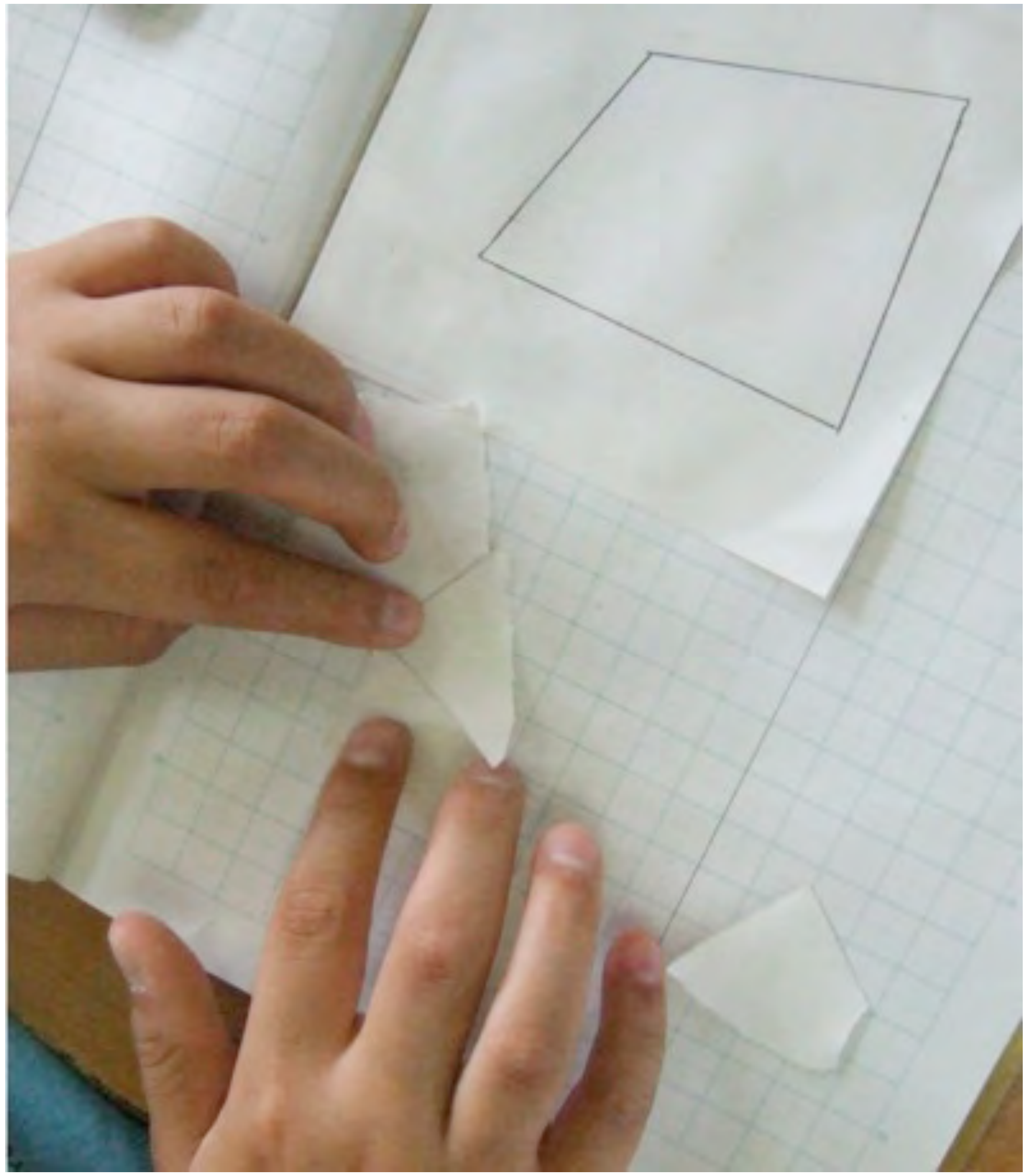
A specific case



Convince yourself.

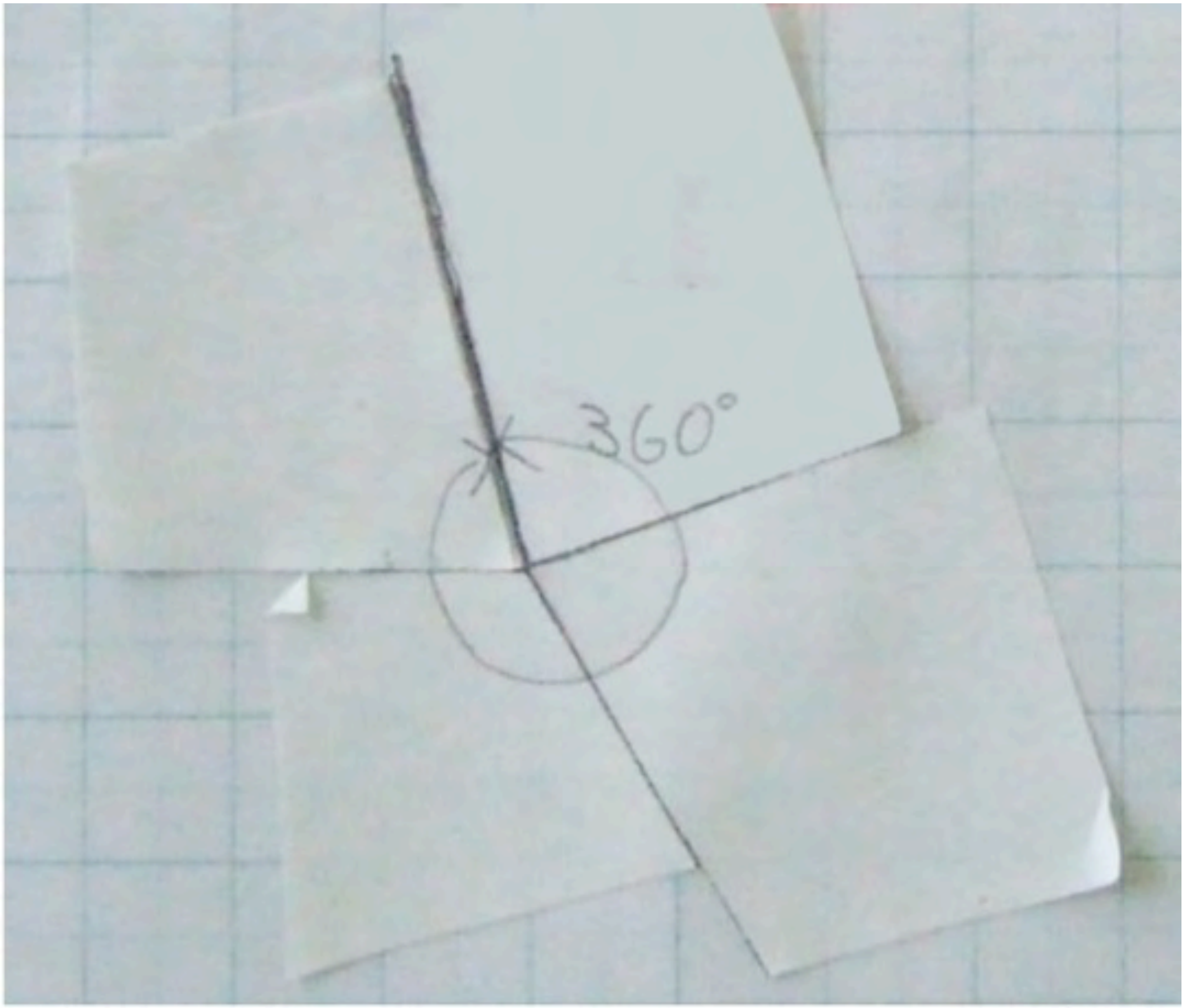
Using methods from the triangle

Gathering all the angles at a point



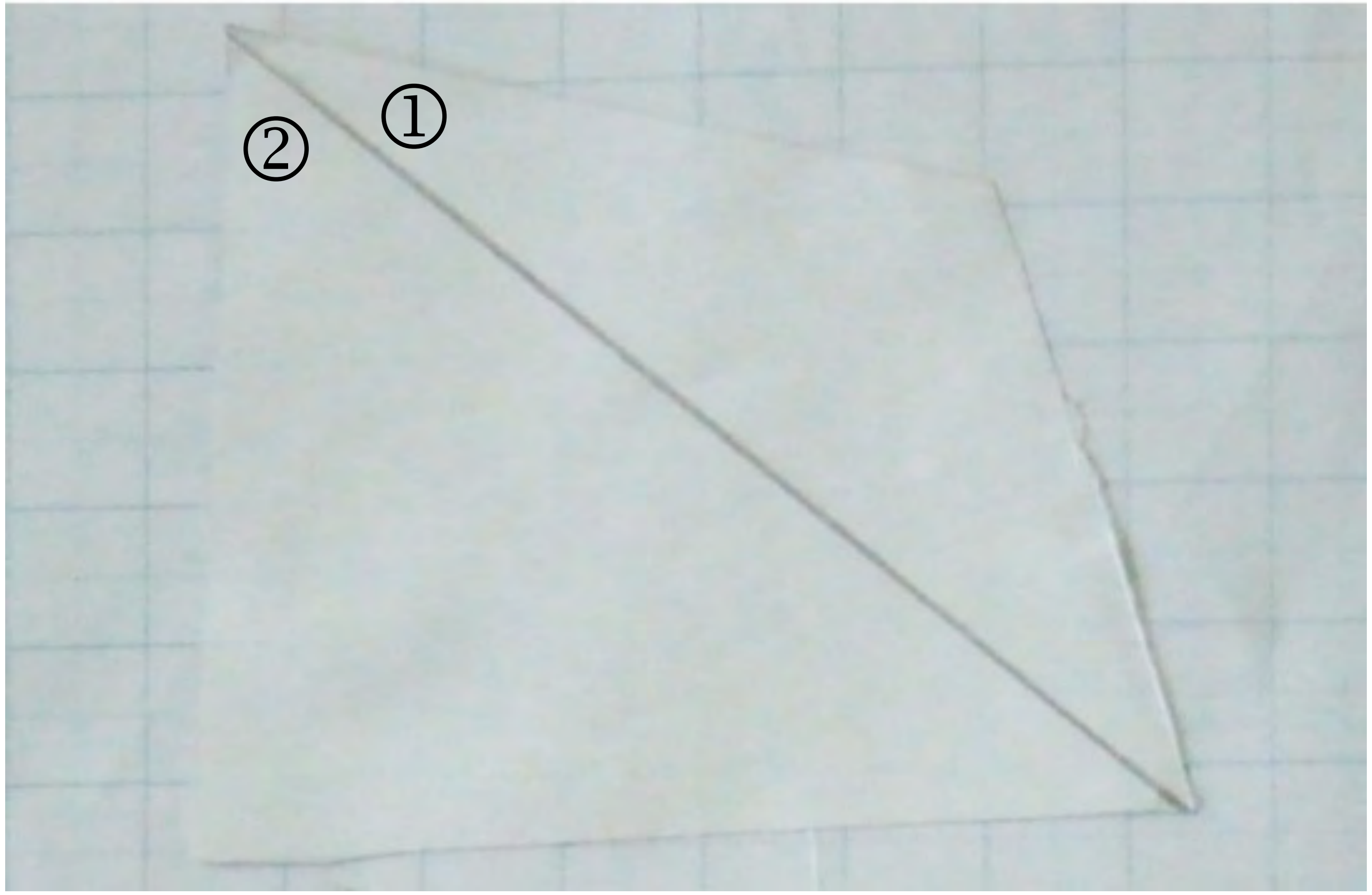
Using methods from the triangle

Gathering all the angles at a point



Dividing into two triangles

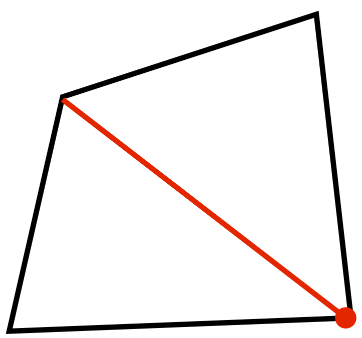
adjacent angles



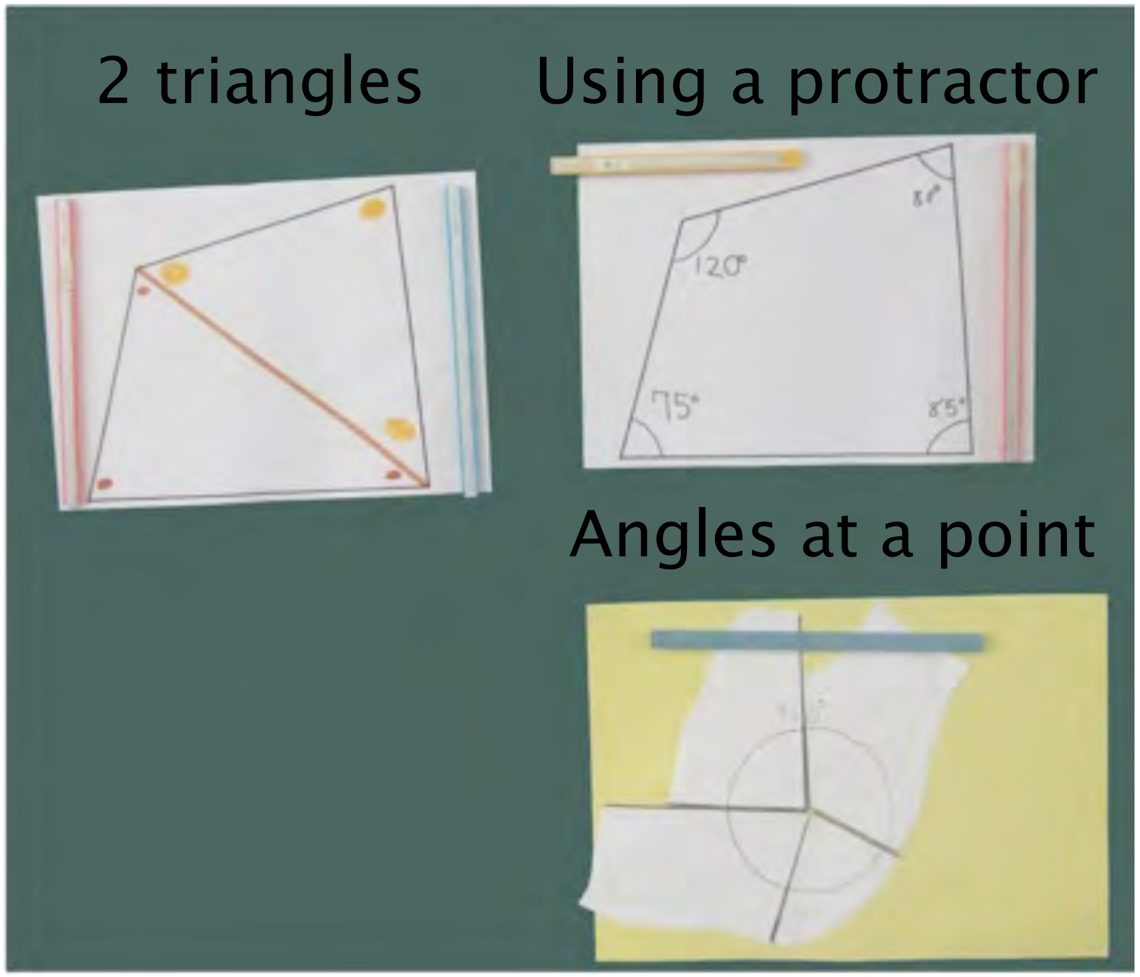
Convince
a sceptic.

Angle sum of a quadrilateral

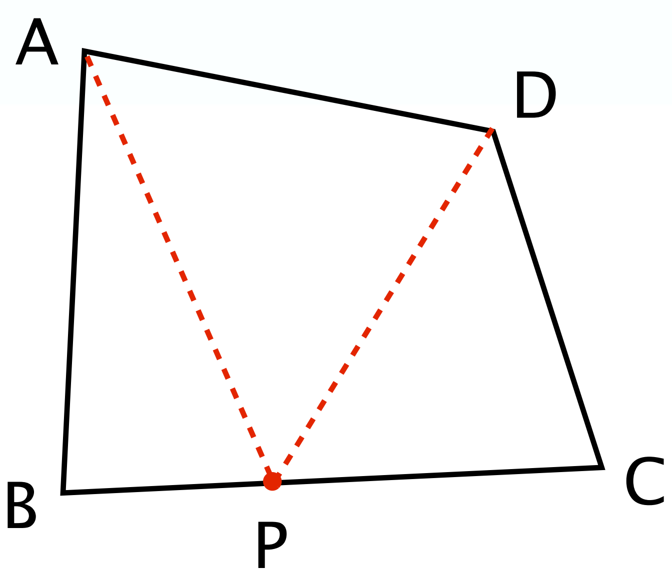
Demonstrate that the angle sum of a triangle is 180˚ and use this to find the angle sum of a quadrilateral. (ACMMG166; Stage 4)



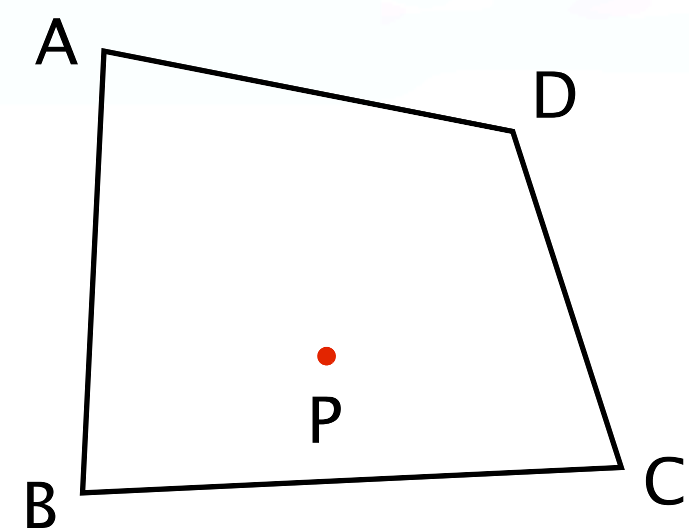
Vertex point



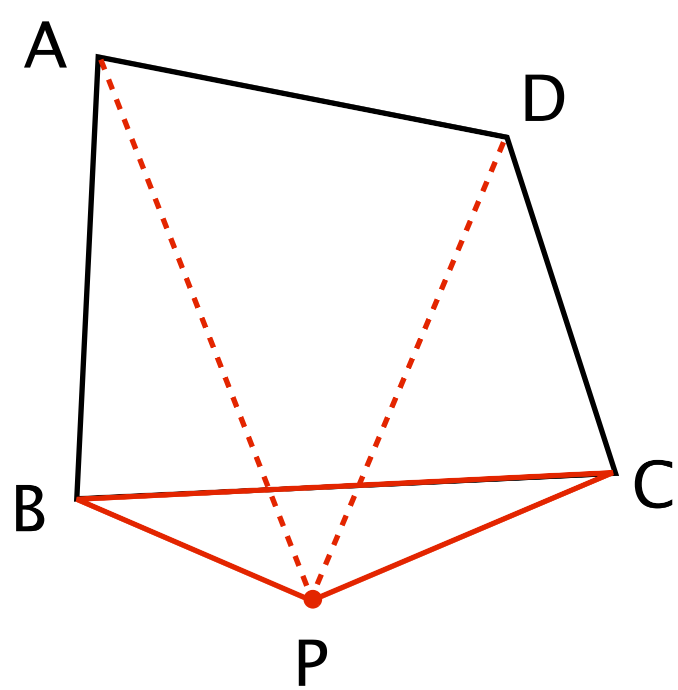
Angle sum of a quadrilateral



point on the edge



internal point

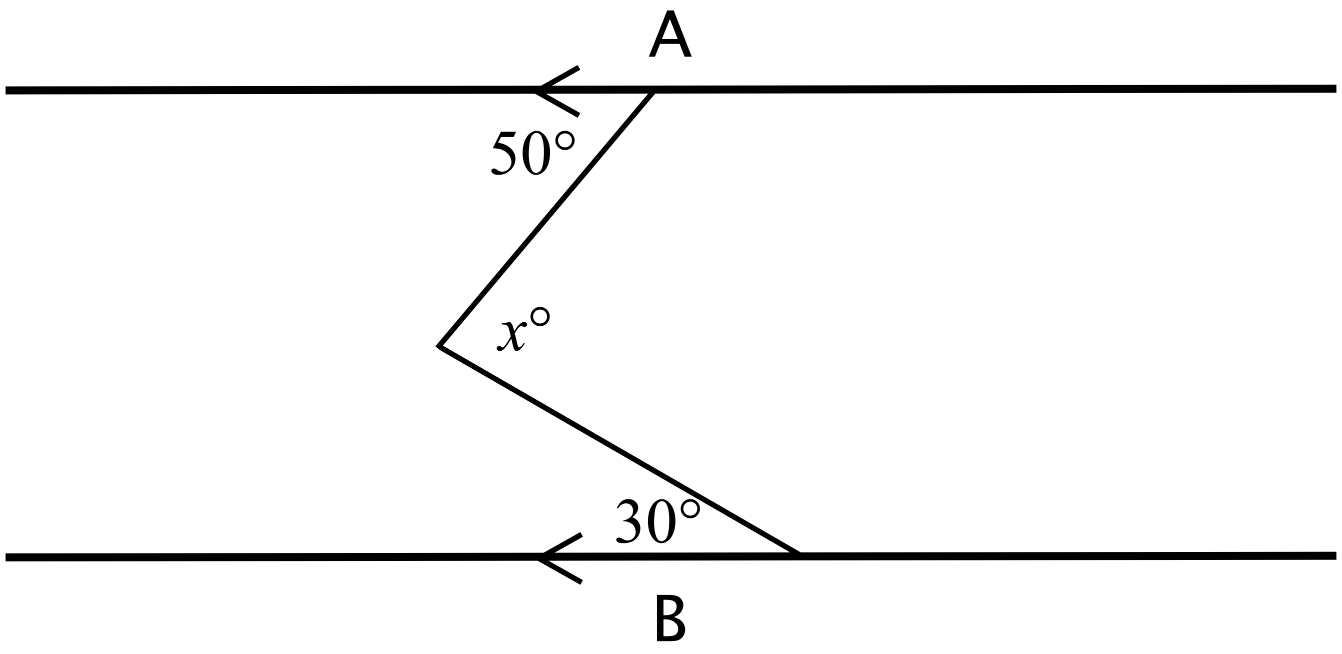


external point

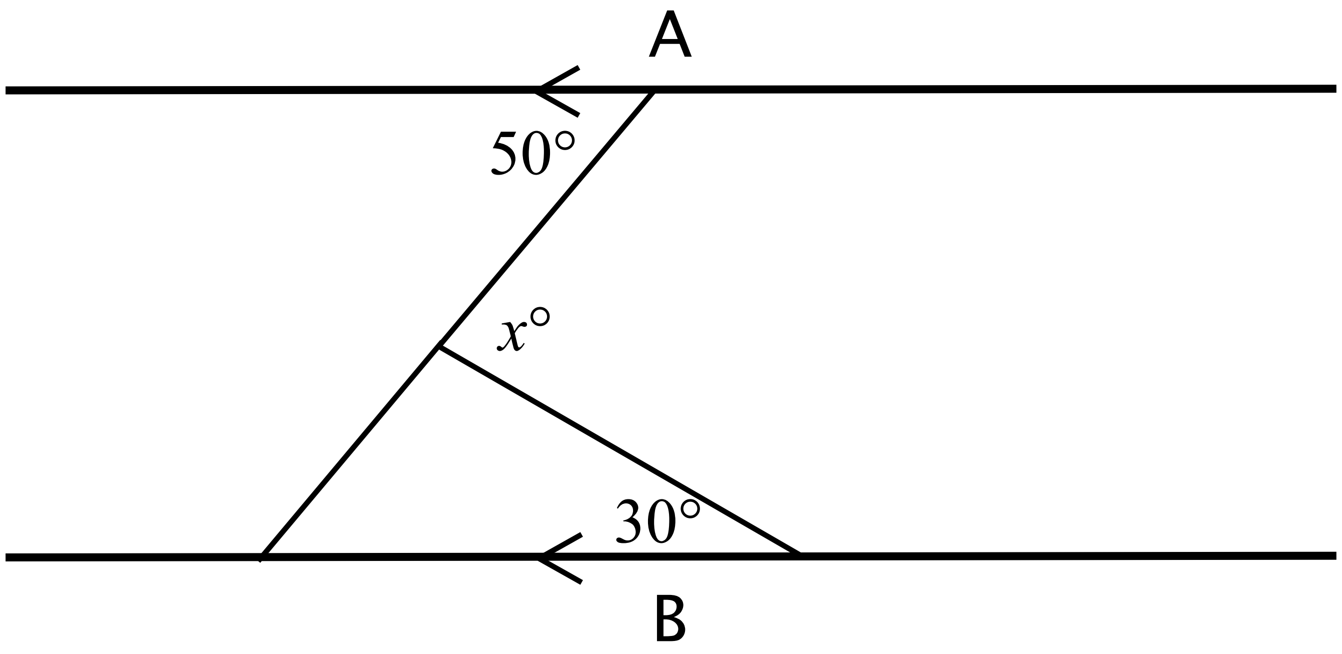
More than one pathway!

Angle in the bend

What do you need to do to be able to answer this question?

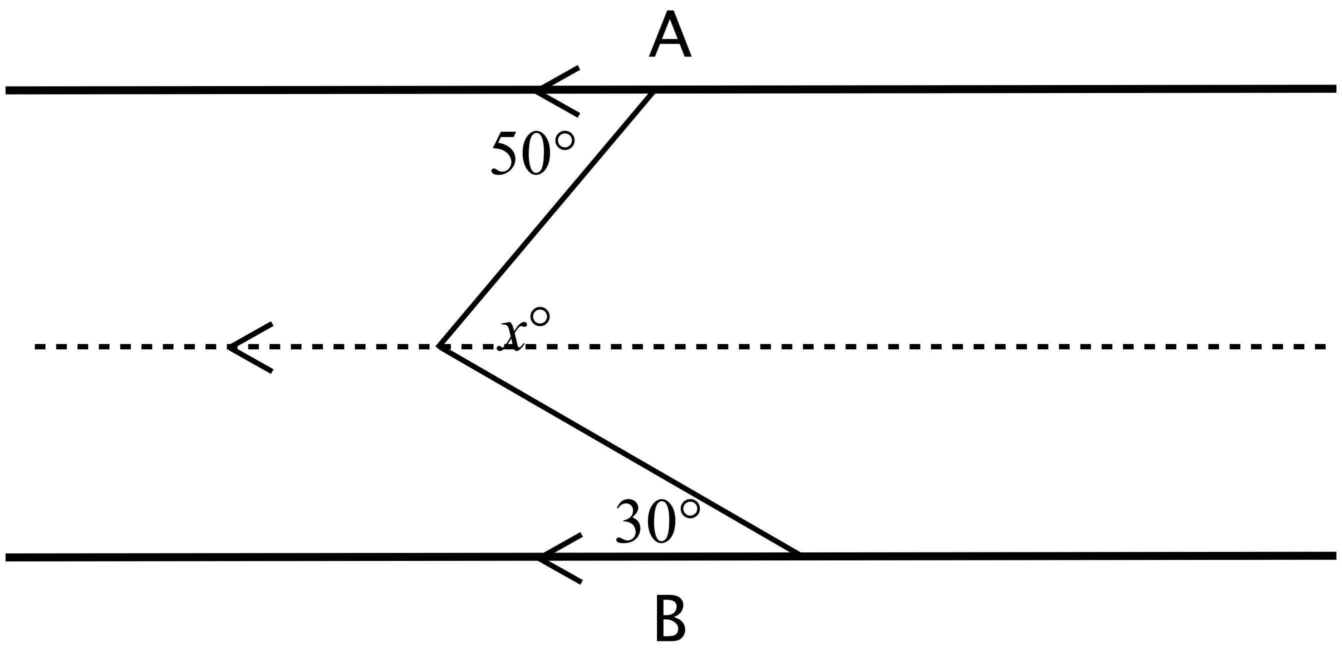


Add a line



Where else?

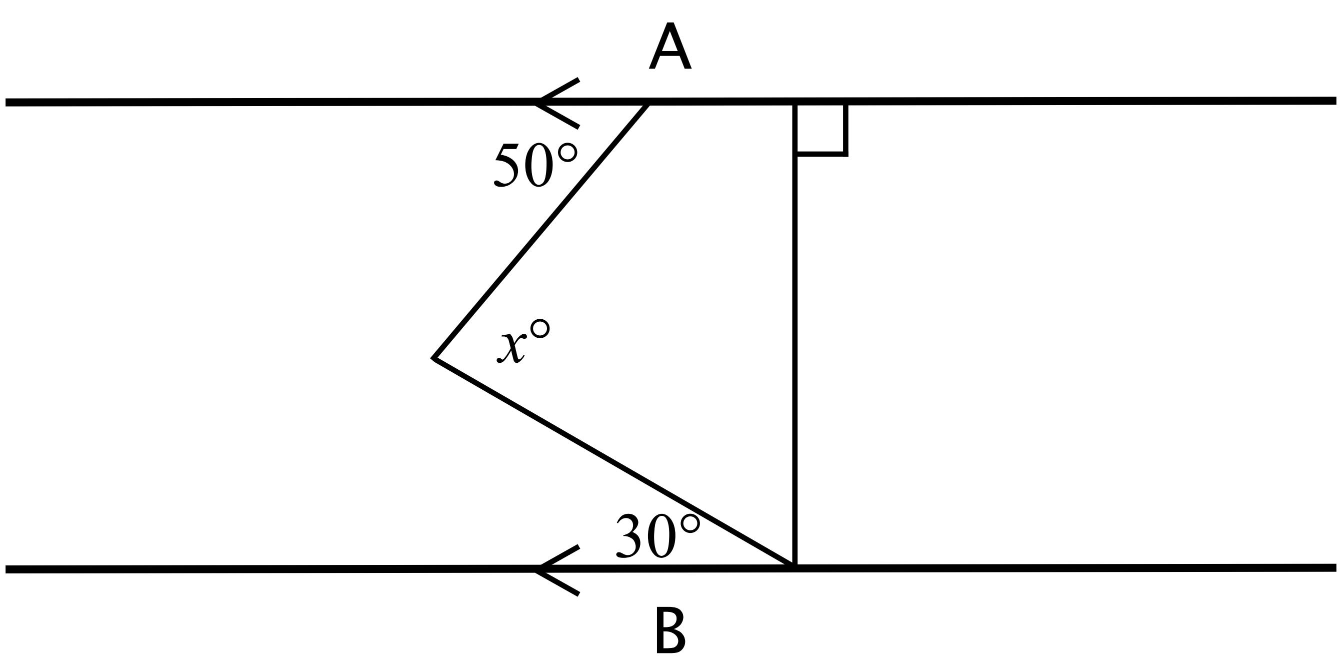
Adding a line



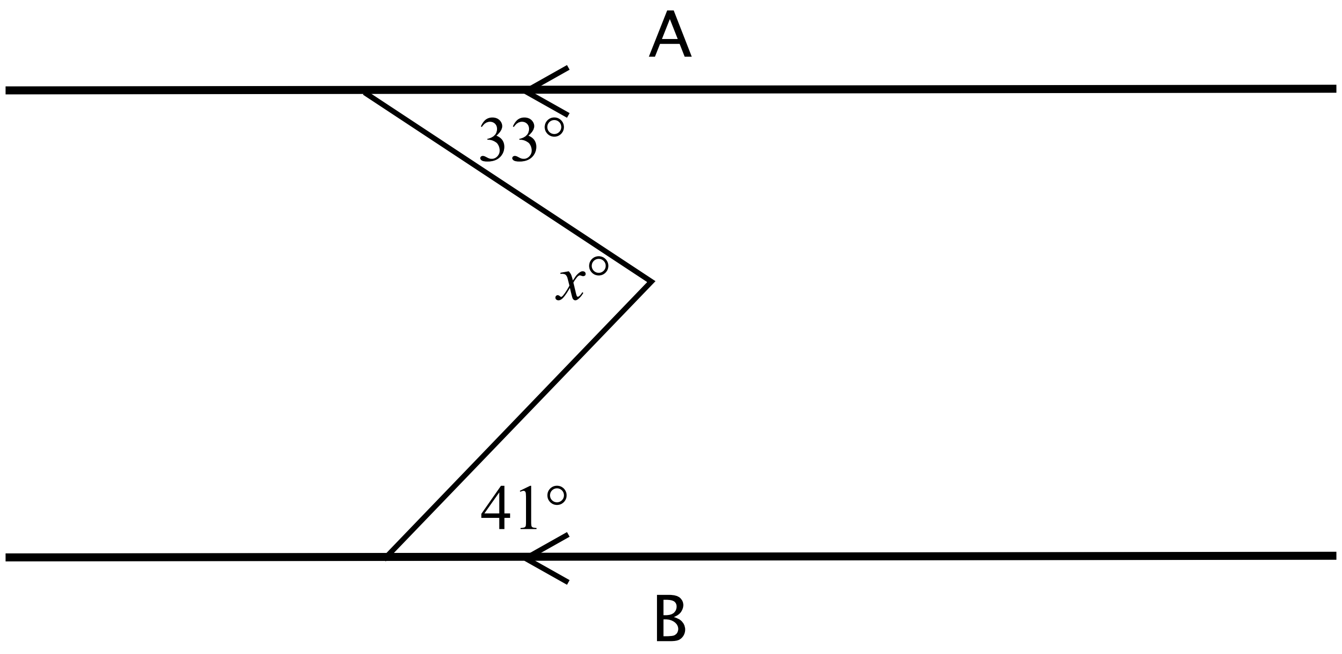
Where else?

Adding a line

Angle sum of a quadrilateral



One to try



Engaging students

"…when individuals engage in tasks in which they are motivated intrinsically they tend to exhibit a number of … desirable behaviours including increased time on task, persistence in the face of failure, more elaborate processing, the monitoring of comprehension,

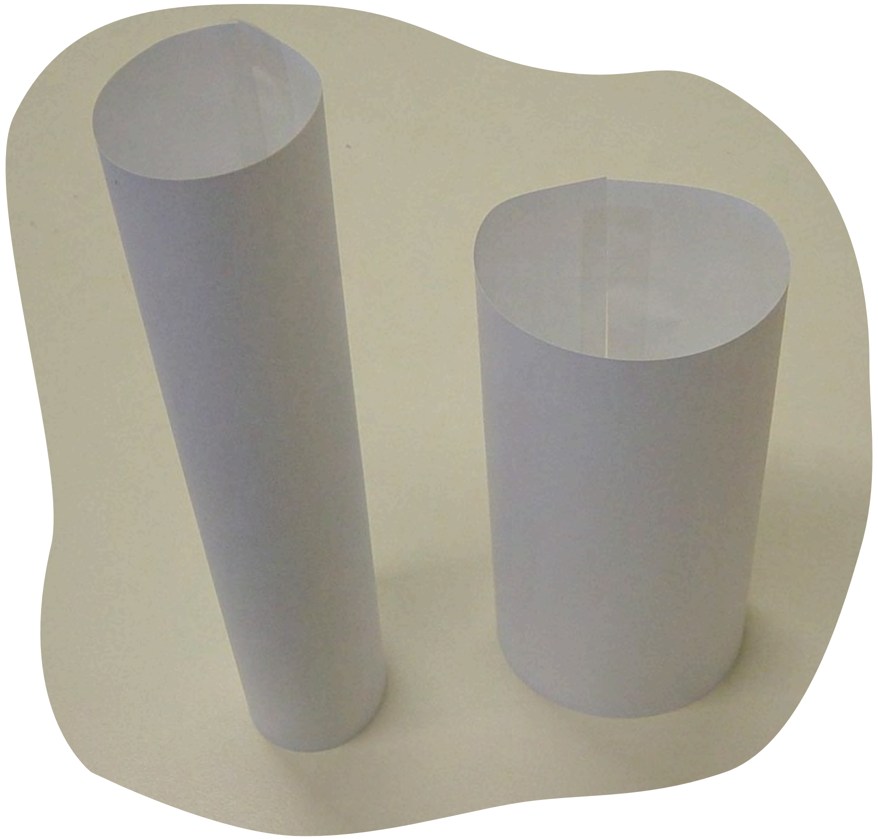
Motivation

… and selection of more difficult tasks, greater creativity and risk taking, selection of deeper and more efficient performance and learning strategies, and choice of activity in the absence of extrinsic reward.” p. 66

Middleton, J. A., & Spanais, P. A. (1999). Motivation for achievement in mathematics: Findings, generalisations and criticisms of the research. *Journal for Research in Mathematics Education*, 30(1) 65-88.

Setting up the problem

Will a (closed) cylinder formed by rolling a piece of paper in landscape orientation have a different capacity from rolling the same piece of paper in portrait orientation?



Measurement reasoning

When we asked Stage 5 students this question as a practical task we obtained a wide range of responses.

Portrait

① measured the height before making the cylinder. (29.7cm) A4

② stick it together

③ flaten the cylinder. !!

④ measure across the top of the cylinder

⑤ calculate the radius.

How do you practically determine the diameter or area of the circular base?

Measurement reasoning

LANDSCAPE

Working out

circumfrence $\div 6.3 =$ radius

2π (1 d.p.)

$296 \div 6.3 = 47$

radius = 47mm
height = 210mm

$\pi \times 47^2 \times 210 = 1457353.416\ mm^3$

significant figures

- The reason we chose to find the radius the way we did is because we think that it is the most accurate procedure to lead us to the answer.

- We then used the formular volume = $\pi r^2 h$

Measurement reasoning

After calculating both volumes as producing very different numeric values, one student recorded that both volumes were the same.

He did, however, comment that the results "*…didn't really make sense*".

We thought the answers would calculate to be the same but they were heaps different.

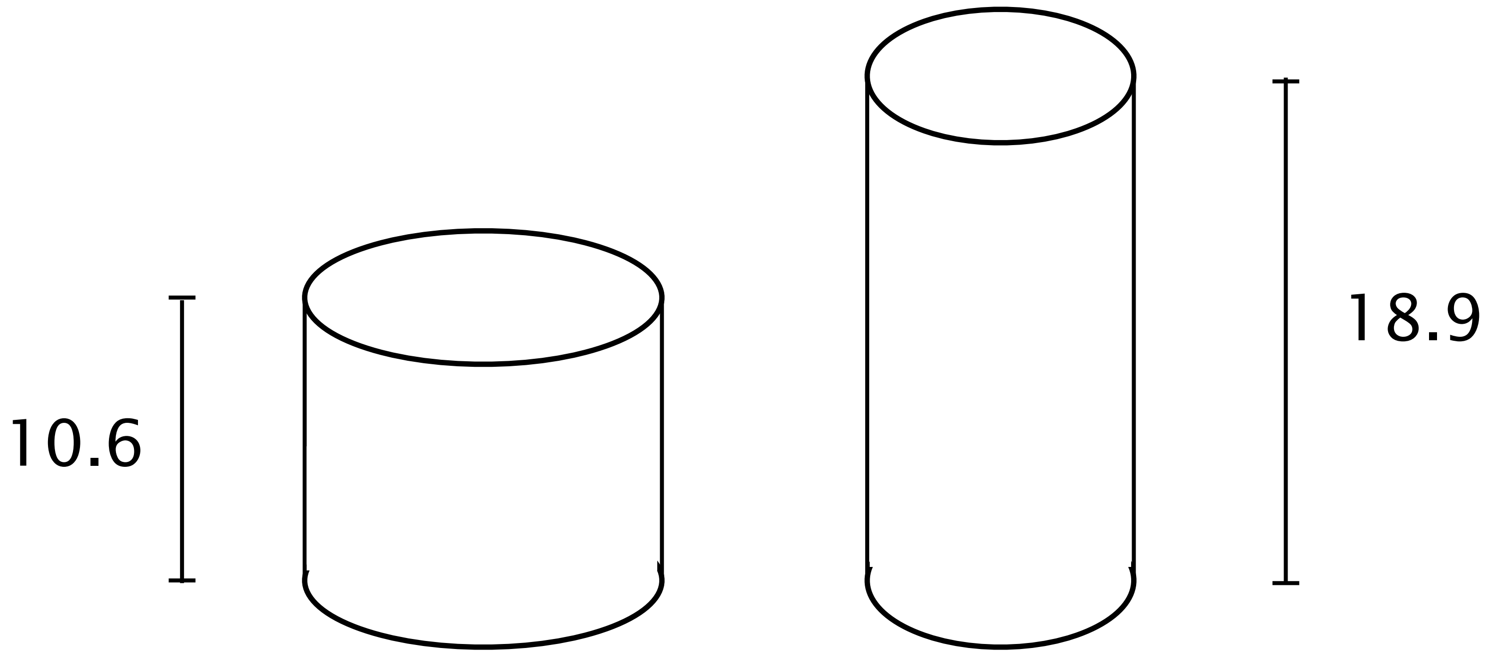
Two Year 10 students

Super-size me

When using a short (10.6 cm), wide glass, adolescents served themselves and **consumed 75% more juice** than a taller (18.9 cm) narrower glass *of the same volume capacity*. Why?

(Fisher & Kral, 2008, p. 43 Super-size me: Portion size effects on young children's eating. *Physiology & Behavior, 94*)

Super-size me



Cylindrical glasses?

McDonald's standard cups

Medium cups: same or different?

For prismoids with planar side-faces

Average end-area rule:
$V = \frac{h}{2}(A_1 + A_2)$

675 mL

500 mL

How would you estimate the capacity?

Prismoidal rule:
$V = \frac{h}{6}(A_1 + 4A_m + A_2)$

Around 1890 BC
Moscow papyrus #14

use formulas to solve problems involving volume (ACMMG198)

Using the end-area rule

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Going beyond the information given

Teaching students to ***reason*** in mathematics relies on students:

1. Understanding what constitutes a valid argument
2. Appreciating there can be more than one path to the answer
3. Recognising you may need to reshape the question to use what you already know.

Going beyond the information given

Teaching students to ***communicate their reasoning*** in mathematics relies on students:

1. Convincing themselves
2. Convincing a friend
3. Convincing an enemy

Solving a mathematics problem involves more than locating the answer in the question.

Mathematical reasoning is more than comprehension.

Going beyond the information given

Mathematical reasoning is more than comprehension.

Everyone else would climb a peak by looking for a path somewhere on the mountain.
Nash would climb another mountain altogether and from that distant peak would shine a searchlight back on the first peak.

A Beautiful Mind, S. Nasar, 1998