Nonparametric Tests for Randomized Block Data with Ties and Ordered Alternatives

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Abstract

Umbrella, Page and Friedman tests are defined and discussed for randomized block designs. The data may be tied. Two alternative derivations of the Page and umbrella test statistics are given. Two sensory evaluation examples are considered.

Keywords: Binary responses, orthogonal contrasts, ranks, sensory evaluation

1. Introduction

Test statistics for ranked and possibly tied data from randomized blocks are presented. Section 2 gives formulae for umbrella and Page tests while Section 3 outlines two approaches for deriving the umbrella and Page test statistics. Section 4 gives another example. Umbrella and Page tests are appropriate when the products to be compared have an ordering associated with them.

Table 1. Ranks within tasters of four lemonades

Taster	A	B	C	D	Sum
1	3	2	1	4	10
2	3	1.5	1.5	4	10
3	1	4	2	3	10
4	3	2	1	4	10
5	4	2	2	2	10
Sum	14	11.5	7.5	17	50

We will illustrate the application of the test statistics based on some data of Sprent and Smeeton (2004, p.233). Suppose lemonades A, B, C and D are the same except for increasing sugar levels of 11%, 12%, 13% and 14%. The ranks for five tasters are given in Table 1. Our other example in section 4 will also involve sensory evaluation.

2. The Umbrella and Page Statistics

Suppose we wish to compare t products or treatments and we have b blocks. Let Y_{ij} be a possibly unobserved score for block i and product j and suppose the model $Y_{ij} = \mu + \beta_i + \tau_j + \varepsilon_{ij}$ applies, where μ is an overall effect, the τ_j are product effects, the β_i are block effects and the ε_{ij} are independent and identically distributed random variables with mean zero. An umbrella statistic, U say, tests the null hypothesis H_0: $\tau_1 = \tau_2 = ... = \tau_t$ against one of the alternatives K_1: $\tau_1 \le \tau_2 \le ... \le \tau_m \ge \tau_{m+1} \ge ... \ge \tau_t$ or K_2: $\tau_1 \ge \tau_2 \ge ... \ge \tau_m \le \tau_{m+1} \le ... \le \tau_t$. Here we assume that m is not known, and that at least one of the inequalities is strict. Moreover depending on which of K_1 or K_2 is specified, p-values for U are either $P(U \ge u)$ or $P(U \le u)$ where u is the observed value of U.

Early nonparametric work on umbrella tests by Mack and Wolfe (1981) was for the completely randomized design and here we extend the use of umbrella tests to randomized blocks.

Table 2 counts how many times each ranking occurs in Table 1. As above, if we consider b blocks (tasters in Table 1) and t products then we can say the probability of each ranking is the count in Table 2 divided by bt.

Table 2. Counts of each ranking in Table 1

Rank (r_s)	1	1.5	2	3	4	Sum
Count (c_s)	3	2	6	4	5	20

In Table 2 r_s, s = 1, …, q denotes the sth ranking and c_s the associated count. The variance of a rank in Table 1 is

$$V = \left\{\sum_{s=1}^{q} r_s^2 c_s / (bt)\right\} - (t+1)^2/4,$$

where q is the number of different values of s; q = 5 in Table 2. Notice that if there were no tied rankings then $c_s/(bt)$ is $1/t$ and $V = (t^2 - 1)/12$. Let

$$\kappa^2 = \{180(t-1)\}/\{bktV(t^2-1)(t^2-4)\}$$

and

$$M = bt(t+2)(t+1)^2/12.$$

Now define the umbrella statistic

$$U = \kappa\left\{M + \sum_{j=1}^{t} j(j-t-1)R_j\right\}.$$

Note that t = 4 here and R_j is the sum of the ranks for the jth product (see the last row of Table 1). This formula for the U test statistic is new. It would be of interest to conduct a power study comparing the U test with other suggested umbrella tests such as that of Kim and Kim (1992).

Another statistic we could calculate for the Table 1 rankings and which is also concerned with ordered alternatives is

$$L^* = \sqrt{\frac{12}{bt^2V(t+1)}}\left\{\left[\sum_{j=1}^{t} jR_j\right] - \frac{bt(t+1)^2}{4}\right\},$$

which is a version of the well-known Page test statistic adjusted for ties. Conover (1998), for example, discusses the Page test for data without ties. Pirie (1985) gives an alternative formula for L^*. Section 3 indicates how L^* and U may be derived. The null hypothesis for Page's test is H_0 above and the alternative is one of K_1 or K_2 above where now m takes the value t.

It can be shown that both U and L^* approximately have the standard normal distribution. Further, both L^* and U can be calculated as components of Pearson's X^2 for an ordinal by ordinal contingency table as illustrated in Rayner et al. (2005, section 5.4.2). We suggest the present approach is more succinct.

For the Table 1 data we find U = 2.19 and L^* = 0.41. An approximate p-value for U is 0.01 and for L^* is 0.34. More exact p-values could be obtained using Monte Carlo methods such as those available in the StatXact software. It appears there is an umbrella effect with the optimum lemonade being C.

For completeness we can also calculate Friedman's well known test statistic adjusted for ties. If we call this F then in our notation

$$F = \left\{\frac{(t-1)}{t}\right\}\left\{\frac{1}{bV}\right\}\left\{\sum_{j=1}^{t}\left[R_j - \frac{b(t+1)}{2}\right]^2\right\}.$$

For the lemonade data F = 6.47 and using the χ_3^2 approximation a p-value is 0.09. This agrees with Sprent and Smeeton (2004) who give an alternative formula for F. As is well known, for the F test H_0 is as above, while the alternative is K_3: $\tau_j \neq \tau_{j'}$ for at least one (j, j') pair. A p-value for an observed value of f of F is $P(F \geq f)$.

3. Derivation of the Page and Umbrella Test Statistics

We outline two approaches.

First, if there is to be a monotonic ordering of rank sums, R_j for j = 1, …, t in a randomized complete block design, then there should be a significant covariance between j and R_j. This explains the form of the Page test statistic $S = \sum_{j=1}^{t} jR_j$.

For randomized block designs first observe that because we use midranks for tied rankings the mean ranking for each block is always $\mu = (t+1)/2$ no matter what ties occur. Let r_{ij} be the rank of the jth product on block i. Then $\mu = \mathrm{E}[r_{ij}]$. Again as above let the probability of getting rank s be $p_s = c_s/(bt)$. Then $V = \mathrm{var}(r_{ij}) = \sum_{s=1}^{q} r_s^2 p_s - \mu^2$. As in section 2 if there are no ties $V = (t^2 - 1)/12$. If $s \neq s'$ then the joint probability of having ranks s and s' is $p_{ss'} = \{c_s/(bt)\}\{c_{s'}/(bt - c_s)\}$ and $\text{cov}(r_{is}, r_{is'}) = \sum_{s \neq s'}^{q} r_s r_{s'} p_{ss'} - \mu^2$. If there are no ties $\text{cov}(r_{is}, r_{is'}) = -(t + 1)/12$. Now

$$\text{E}[S] = \sum_{j=1}^{t} j \sum_{i=1}^{b} E[r_{ij}]$$

$$= \{b(t+1)/2\} \sum_{j=1}^{t} j = bt(t+1)^2/4$$

and

$$\text{var}(S) =$$
$$b\{\sum_{s=1}^{q} r_s^2 \,\text{var}(r_{is}) + \sum_{s \neq s'} r_s r_{s'} \,\text{cov}(r_{is}, r_{is'})\}.$$

The derivations for the umbrella test statistic are parallel.

Observe that $\{S - \text{E}[S]\}/\sqrt{\text{var}(S)}$ is not asymptotically standard normal. However

$$L^* = \{(t-1)/t\}\{S - \text{E}[S]\}/\sqrt{\text{var}(S)}$$

is. Conover (1998, p.393) discusses this correction for data ranked in blocks.

As an alternative to the approach just given consider, for $t \geq 3$, partitioning the Friedman test statistic using orthogonal *trend* contrasts. Orthogonal contrasts for planned comparison of means are often given in introductory statistics texts. See, for example, Moore et al. (2009, section 12.2). Orthogonal trend contrasts are similar and are also well known. See, for example, Kuehl (2000, section 3.3). Let $\overline{R}_j = R_j/b$. If

$$L = \sum_{j=1}^{t} \lambda_j \overline{R}_j \text{ and } Q = \sum_{j=1}^{t} \pi_j \overline{R}_j$$

are the first and second orthogonal trend contrasts then

$$(L^*)^2 = \frac{CL^2}{\sum_{j=1}^{t} \lambda_j^2} \text{ and } U^2 = \frac{CQ^2}{\sum_{j=1}^{t} \pi_j^2}$$

where $C = b(t - 1)/(tV)$. If there are no ties then $C = 12b/\{t(t + 1)$. For $t > 3$ the partition of F can be completed by difference.

Orthogonal trend contrasts could similarly be used to partition the Kruskal-Walls statistic for one-way layout data and the Durbin statistic for balanced incomplete block data. The orthogonal trend contrasts can also be used to obtain powerful goodness of fit tests for the discrete uniform distribution with ordered categories, by partitioning Pearson's X^2 statistic. See Rayner et al. (2009, section 5.4).

The Appendix lists λ and π values for t = 3, 4, 5 and 6. Executable code for the Command Prompt in Microsoft Windows is available from the first author.

4. Another Example

We now consider some binary response data where it is slightly less obvious that the statistics above apply.

Suppose milk has been refrigerated for five days in four containers: opaque plastic (A), cardboard (B), clear plastic (C) and glass (D). Six judges were asked "is the milk fresh?" The data are given in Table 3 where Y = 'yes' and N = 'no'. The containers are ordered in terms of the amount of light transmitted.

Table 3. Responses to milk storage trial

Judge	A	B	C	D
1	Y	Y	Y	N
2	Y	Y	Y	N
3	Y	Y	N	Y
4	Y	Y	N	Y
5	Y	Y	N	Y
6	Y	Y	N	N

We find $L^* = -2.310$ with a p-value of 0.01, $F = 8.053$ with a p-value of 0.045 and $U = 0.40$ with a p-value of 0.34. It appears there is a strong ordering of the preference for containers and that an opaque container is preferred to keep milk fresh longer.

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Appendix

(a) Linear Coefficients

t	$\lambda_1, \lambda_2, \ldots, \lambda_t$	$\sum_{j=1}^{t} \lambda_j^2$
3	– 1, 0, 1	2
4	– 3, – 1, 1, 3	20
5	– 2, – 1, 0, 1, 2	10
6	– 5, – 3, – 1, 1, 3, 5	70

(b) Quadratic Coefficients

t	$\pi_1, \pi_2, \ldots, \pi_t$	$\sum_{j=1}^{t} \pi_j^2$
3	1, – 2, 1	6
4	1, –1, – 1, 1	4
5	2, –1, – 2, 1, 2	14
6	5, – 1, – 4, – 4, – 1, 5	84